Some Properties of Closed Convex Curves in a Plane.

By Arnold Emch.

1. The main object of this investigation is to prove that at least one square may be inscribed in every closed convex curve in a plane. Instead of "closed convex curve" in the ordinary sense I shall use throughout the equivalent shorter term "oval."

But before the main proposition can be proved it is necessary to give definitions of a convex domain and of an oval in particular, and to establish a number of preliminary theorems.

(1) Minkowsky* has given the following definition for a continuous domain of points enclosed by a convex boundary in a plane:

(1a) The domain contains with any two points also the entire segment between the two points.

(1b) The domain is finite.

(1c) The domain is closed.

The points of the boundary belong to the domain. Through every point of the boundary there is at least one straight line, so that all points of the domain which are not on the line lie on one side of the line only. Minkowsky calls such a line a "supporting line" (*Stützlinie*).

(1d) An *oval* in particular encloses such a domain (including the oval) and can be defined parametrically by two distinct continuous single-valued periodic functions

$$x=\phi(t), \qquad y=\psi(t)$$

of a real parameter t and with the common period ω.† The derivatives $\phi'(t)$ and $\psi'(t)$ are also periodic (same period ω) and are assumed continuous for all definite values of t, which merely requires continuity within the interval $0 \leqq t \leqq \omega$. It is also assumed that $\phi'(t)$ and $\psi'(t)$ do not vanish simultaneously for any values of t. Thus, singular points are excluded. We include furthermore in the definition of an oval that for no parts of the period-interval the

* Theorie der konvexen Körper, *Gesammelte Abhandlungen*, Vol. II, p. 154.

† See Osgood, *Lehrbuch der Funktionentheorie*, Vol. I, pp. 120–123.

functions $\phi(t)$ and $\psi(t)$ remain constant or depend linearly upon t. This excludes any straight portions of the boundary.

From this definition (1d), follows that at every point of the oval there exists a definite tangent (including the cases in which for certain values of t $\lim \left\{\frac{\psi'(t)}{\phi'(t)}\right\} = \pm\infty$). If the point of tangency varies continuously, the direction of the tangent varies continuously.

2. For the proof of some of the theorems that follow, we make use of the following properties of continuous single-valued periodic functions as considered under (1d):

By choosing the origin of t properly we can always assume that for $t=0$, or $t=\omega$, none of the functions $\phi(t)$, $\psi(t)$, $\phi'(t)$, $\psi'(t)$ vanish. At the extremities of the period interval there is $\phi(0)=\phi(\omega)$, $\phi'(0)=\phi'(\omega)$, $\psi(0)=\psi(\omega)$, $\psi'(0)=\psi'(\omega)$. Thus, from Rolle's theorem follows as an application

THEOREM I: *If $\phi(t)$ and $\psi(t)$ have real roots within the period-interval, then their number is in each case even. $\phi'(t)$ and $\psi'(t)$ have always at least two and generally an even number of real roots.*

3. Consider now any direction with the slope τ in the plane of the oval. The question is, are there any tangents to the oval parallel to this direction, and if there are any, what is their number? We evidently have the condition $\frac{dy}{dx}=\tau$, or

$$\psi'(t)-\tau\phi'(t)=0.$$

$\psi'(t)-\tau\phi'(t)$ is the derivative of $\psi(t)-\tau\phi(t)$, and as the latter is a periodic function of the prescribed type, also its derivative is coperiodic and consequently admits according to theorem I an even number of roots. Hence

THEOREM II: *There exist at least two tangents to an oval parallel to any given direction.*

To prove the

THEOREM III: *There are always two and only two tangents to an oval parallel to any given direction.*

Suppose that there were three distinct parallel tangents t_1, t_2, t_3, with t_2 between t_1 and t_3. Then, there would be points belonging to the domain on both sides of t_2, which, according to the property of a supporting line, is impossible.

When, in the equation

$$\psi'(t)-\tau\phi'(t)=0,$$

τ changes continuously, then also the two real roots change continuously. Geometrically, this is equivalent to the fact that *when a given direction changes continuously, then each of the two tangents to the oval changes continuously.*

4. THEOREM IV: *No reentrant quadrangle can be inscribed in an oval.*

As a definition I state that in a reentrant quadrangle $A_1A_2A_3A_4$ one of the vertices, say A_4, lies within the boundary of the triangle formed by the remaining three and in which at each vertex two sides of the quadrangle meet, as in Fig. 1. On the oval the vertices of an inscribed reentrant quadrangle are assumed to follow each other in the order $A_1A_4A_3A_2$. According to (1a) all points of the segments A_1A_2 and A_3A_4 are in the domain, while those outside of these segments on their prolongations are outside of the domain. According to the same condition, A_5, the point where the prolongation of A_3A_4 meets A_1A_2, belongs to the domain. Hence, since A_4 and A_5 belong to the domain, all points of the segment A_4A_5 belong to the domain. This, however, is in contradiction to the assertion that all points outside of the segment A_3A_4 are excluded from the domain. A reentrant quadrangle can therefore never have its vertices on an oval.

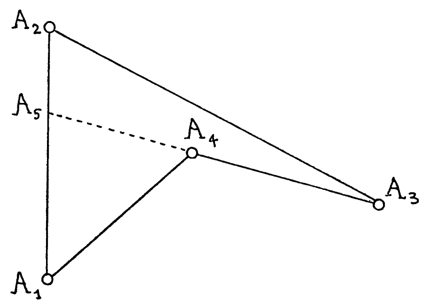

FIG. 1.

THEOREM V: *Two distinct rhombs with corresponding parallel sides or parallel axes can never be inscribed in the same oval.*

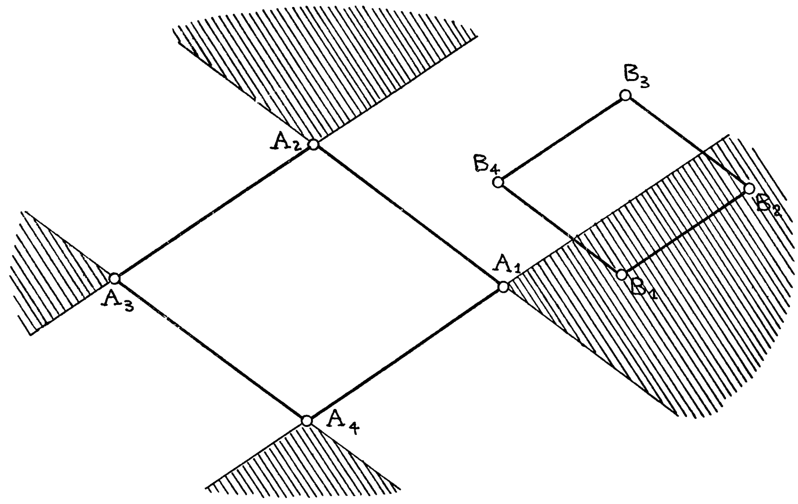

FIG. 2a.

To prove this, assume first that the second rhomb $B_1B_2B_3B_4$ has one vertex, say B_1, in any one of the five shaded regions determined by the first rhomb $A_1A_2A_3A_4$, Fig. 2a. Then there are always three points of the first rhomb which with B_1 form a reentrant quadrangle. The other possibility left for the location of the second rhomb is within the four blank regions of Fig. 2a. In

this second case all vertices of the first rhomb are within the shaded region determined by the second rhombus, Fig. 2b, so that there are always three points B which with any vertex A form a reentrant quadrangle. Both cases include those where points of one rhombus lie on the sides of the other. Hence, no matter what the relative position of the two rhombs may be, there exists always at least one reentrant quadrangle among the eight vertices, and consequently, according to the theorem IV, no oval can pass through them. In a similar manner the proof can be extended without difficulty to figures with parallel axes only.

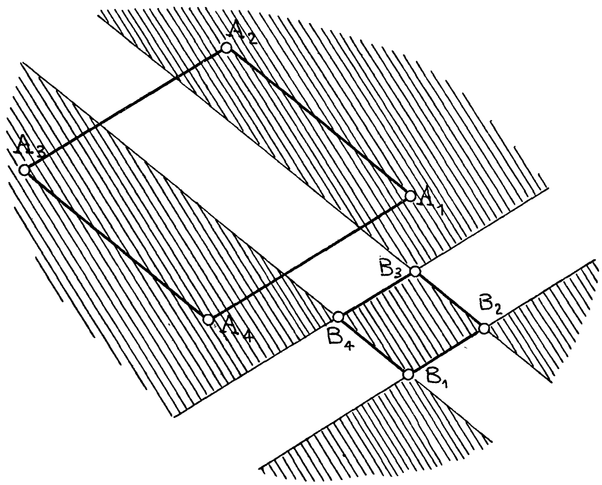

FIG. 2b.

5. In what follows I shall also have to make use of a proposition in function theory which may be stated as follows:

THEOREM VI: *Let a and b be two distinct real numbers, and let $\lambda=\phi_1(\theta)$, $\mu=\psi_1(\theta)$ be two uniform continuous real functions of a parameter θ, subject to the only condition that two distinct values α and β of the parameter θ exist, so that*

$$\phi_1(\alpha)=\psi_1(\beta)=a,$$
$$\phi_1(\beta)=\psi_1(\alpha)=b;$$

then there exists at least one value of θ, say $\theta=\gamma$, for which $\lambda=\mu$, or

$$\phi_1(\gamma)=\psi_1(\gamma).$$

The proof follows immediately from the fact that $\phi_1(\theta)-\psi_1(\theta)$ is continuous and hence takes every value (at least once) between $a-b$ and $b-a$. That is, there is at least one value of θ, $\theta=\gamma$, such that $\phi_1(\gamma)=\psi_1(\gamma)$.

6. It is now possible to prove

THEOREM VII: *It is always possible to inscribe a square in an oval.*

For this purpose assume any point O in the plane of this curve and draw any line l_α through this point, and determine the mid-points of all chords of the oval parallel to l_α and designate the points of tangency of the tangents parallel to l_α by S_α and T_α. The locus of these mid-points is a certain continuous curve C_α, extending from S_α to T_α. Next, draw through O a line $l_\beta \perp l_\alpha$ and repeat the same construction with respect to this direction. The result is a continuous curve C_β extending from S_β to T_β. As the tangents parallel to l_α and l_β form

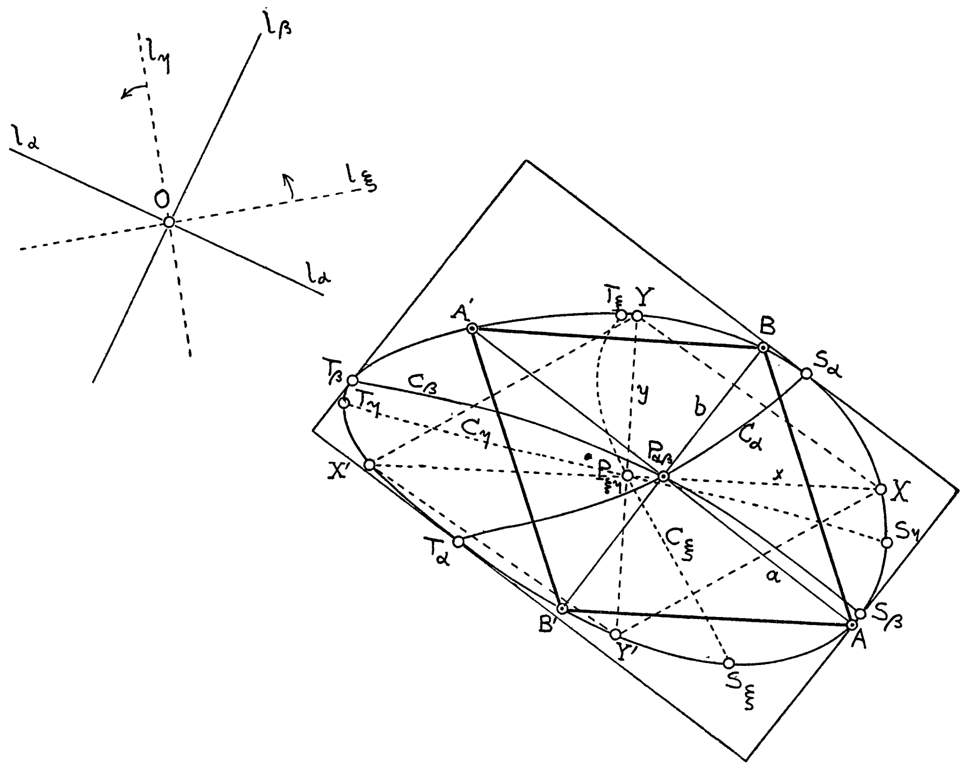

FIG. 3.

a rectangle, it is easily seen that C_α and C_β necessarily intersect within the domain of the oval, Fig. 3. In fact *there is always only one real point of intersection $P_{\alpha\beta}$ between C_α and C_β within the domain.*

To prove this, assume that there are two points of intersection; then there would exist two rhombs with parallel axes inscribed in the oval, which is in contradiction to theorem V. The extremities of the lines through $P_{\alpha\beta}$ parallel to l_α and l_β on the oval form a rhombus $ABA'B'$. Thus, with every pair of orthogonal rays l_α and l_β through O is associated one definite rhombus inscribed in the oval; and the same rhombus is evidently obtained when l_α and l_β are interchanged.

There exists therefore a (1, 1) *correspondence between all pairs of orthogonal lines through O and all rhombs inscribed in the given oval.*

Turning a line l_ξ through O continuously from l_α to l_β, then its orthogonal ray l_η will turn in the same sense from l_β to l_α. The corresponding curves C_ξ and C_η, since their extremities S and T on the oval change continuously, also change continuously. Their point of intersection $P_{\xi\eta}$ describes therefore a continuous curve, and consequently the corresponding rhombus $XYX'Y'$ changes continuously. The axes $\lambda = XX'$ and $\mu = YY'$ of this rhombus may therefore be expressed as uniform and continuous functions of a parameter θ associated with the direction of l_ξ, within the interval between l_α and l_β and including these limits. We may, for instance, choose as θ the positive angle which l_ξ makes with the positive part of the X-axis. Designating the diagonals of the original rhombus by a and b, by α and β the parameters associated with l_α and l_β, by

$$\lambda = \phi(\theta), \qquad \mu = \psi(\theta)$$

the axes of the rhombus as the above uniform and continuous functions of θ within the interval $\alpha \leqq \theta \leqq \beta$, then

$$a = \phi(\alpha), \qquad b = \psi(\alpha).$$

If now the line l_ξ turns from l_α to l_β, $XYX'Y'$ changes from $ABA'B'$ to $BA'B'A$, so that in the second position

$$\lambda = \phi(\beta) = b, \qquad \mu = \psi(\beta) = a.$$

Hence, the situation is exactly as stated in theorem IV. There exists therefore at least one direction, l_γ, for which $\phi(\gamma) = \psi(\gamma)$, or $\lambda = \mu$; *i. e.*, where the rhombus becomes a square.